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Dimension Spectrum of Axiom A Diffeomorphisms. I. The Bowen–Margulis Measure

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We compute the dimension spectrum $f(\alpha)$ of the singularity sets of the Bowen–Margulis measure defined on a two-dimensional compact manifold and invariant with respect to a C^2 Axiom A diffeomorphism. It is proved that f is the Legendre–Fenchel transform of a free energy function which is real analytic (linear in the degenerate case). The function f is also real analytic on its definition domain (defined in one point in the degenerate case) and is related to the Hausdorff dimensions of Gibbs measures singular with respect to each other and whose supports are the singularity sets, and we decompose these sets.

KEY WORDS: Multifractal; thermodynamic formalism; Hausdorff dimension; free energy function; large deviations; Gibbs measures.

INTRODUCTION

Let (X, μ, g) be a dynamical system, where X is a metric compact space, g a transformation onto X, and μ a g-invariant measure on X. Multifractal analysis is concerned with the decay rates of the measures $\mu(U)$ where $|U|$ goes to 0 ($|U|$ denotes the diameter). To this purpose we define the maps

$$\alpha^{+}(x) = \overline{\lim_{\substack{x \in \mathrm{int}(U) \\ |U| \to 0}}} \frac{\mathrm{Log}\, \mu(U)}{\mathrm{Log}\, |U|} \qquad \text{and} \qquad \alpha^{-}(x) = \underline{\lim_{\substack{x \in \mathrm{int}(U) \\ |U| \to 0}}} \frac{\mathrm{Log}\, \mu(U)}{\mathrm{Log}\, |U|} \tag{0.1}$$

which lead to the definition of the singularities of the measure μ in one point: when we have $\alpha^{+}(x) = \alpha^{-}(x) = \alpha(x)$, then $\alpha(x)$ represents a local dimension [μ has pointwise dimension $\alpha(x)$] and we write $\mu(U) \sim |U|^{\alpha(x)}$. This notion was introduced by Frostman in potential theory with the

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capacities, and we can develop further this study for invariant measures of dynamical systems.

Some theoretical physicists[7] have found relevant information in the singularity sets

$$C_\alpha^+ = \{x/\alpha^+(x) = \alpha\}, \qquad C_\alpha^- = \{x/\alpha^-(x) = \alpha\}, \qquad C_\alpha = C_\alpha^+ \cap C_\alpha^- \tag{0.2}$$

and using thermodynamic formalism, they obtain results about local singularities of a measure. Most of them concern expanding hyperbolic dynamical systems when g is C^2 (or $C^{1+\delta}$) and $g' \geqslant \gamma > 1$: $\dim X = 1$ ($X = [0; 1]$ or S^1),[2,17] and $\dim X = 2$ ($X = [0; 1]^2$ or $\mathbb{T}$)[13,23] when μ has nonzero Lyapunov exponents. There also exist results for local singularities for a class of random measures (multiplicative chaos) obtained by random iterated multiplications.[3,6] They correspond to a rigorous study of the phase transition of a system with random interactions.

When the measure μ is ergodic, then there exists a real $\alpha > 0$ such that

$$\alpha(x) = \alpha \quad \mu\text{-a.e.} \qquad \text{and} \qquad \mu(C_\alpha) = 1$$

It is then interesting to study the singularity sets (0.2) when they are not empty when α^+ and α^- take different values. We then obtain fractals, and in order to recognize them, we define the dimension spectrum function $f(\alpha)$:

$$f(\alpha) = HD(C_\alpha^\pm) \qquad \text{and} \qquad f \equiv -\infty \quad \text{when the sets are empty} \tag{0.3}$$

Using large-deviations results, it is easy to prove the inequality $HD(C_\alpha^\pm) \leqslant f(\alpha)$. To prove the reverse inequality, $HD(C_\alpha^\pm) \geqslant f(\alpha)$, we apply a Frostman's lemma to a measure constructed on a set $V_\alpha \subset C_\alpha$ (this construction is recursive and depends on several appropriate sequences).

Under suitable assumptions this function f is the Legendre–Fenchel transform of a free energy function F concave and C^1

$$f(\alpha) = \inf_{t \in \mathbb{R}} \{t\alpha - F(t)\} \tag{0.4}$$

where F is defined from a sequence of partition functions $(Z_n)_{n \geqslant 1}$

$$\forall \beta \in \mathbb{R}, \quad F(\beta) = \lim_{n \to +\infty} -\frac{1}{n} \operatorname{Log} Z_n(\beta) \tag{0.5}$$

with

$$Z_n(\beta) = \sum_{\substack{U \in U_n \\ \mu(U) > 0}} \mu(U)^\beta$$

where $(U_n)_{n \geqslant 1}$ is a partition whose diameter tends to 0 when n tends to $+\infty$.

The Model

We take X to be a compact manifold of dimension 2 (for example, the torus $\mathbb{T}$) and g is a C^2 Axiom A diffeomorphism. The g-invariant measure μ is the Bowen–Margulis measure, the one that realizes the maximum of topological entropy. See also ref. 16, where an analogous example is treated.

In order to prove our main result

$$f(\alpha) = HD(C_\alpha) \quad \text{for} \quad \alpha \in [\alpha_1 ; \alpha_2] \subset \mathbb{R}^{+*} \qquad \text{and} \qquad f \equiv -\infty \quad \text{otherwise}$$

and f is real analytic on $]\alpha_1 ; \alpha_2[$ (in the degenerate case $\alpha_1 = \alpha_2$), we are going to prove the existence with explicit formulas, the regularity, and some other properties of a free energy function F related to the dimension spectrum f as in (0.4).

1. DEFINITIONS, NOTATIONS, AND PRELIMINARIES

The tangent space to X can be written

$$T_X = \bigcup_{x \in X} T_x \quad \text{(tangent space at the point } x\text{)}$$

and we represent the differential map of g at x by $Dg_x \colon T_x \to T_{g(x)}$.

Definition 1.1. A set Γ is said to be *hyperbolic* if[1, 18]:

- Γ is closed and $g(\Gamma) = \Gamma$.
- $\forall x \in \Gamma$, $T_x = E^u_x \oplus E^s_x$ with $Dg(E^u_x) = E^u_{g(x)}$ and $Dg(E^s_x) = E^s_{g(x)}$.
- $\exists c > 0$ and $\lambda \in]0; 1[$ such that for any integer n we have

$$\forall (v, w) \in E^u_x \times E^s_x, \quad \|Dg^n(v)\| \leqslant c\lambda^n \|v\| \qquad \text{and} \qquad \|Dg^{-n}(w)\| \leqslant c\lambda^n \|w\|$$

- E^u_x and E^s_x vary continuously with x.

Definition 1.2. A point $x \in X$ is *nonwandering* if

$$\forall V \in \mathscr{V}_{(x)}, \quad V \cap \left(\bigcup_{n \geqslant 1} g^n(V) \right) \neq \varnothing$$

Let $\Omega = \Omega(g) = \{x \in X / x \text{ nonwandering}\}$. The set Ω is closed, g-invariant, and $\{x \in X / x \text{ periodic}\} \subset \Omega$.

Definition 1.3. g is said to be Axiom A if and only if Ω is hyperbolic and $\overline{\{x \in X/x \text{ periodic}\}} = \Omega$ (g is Anosov if X is hyperbolic). We define the sets which are the stable manifolds (respectively unstable) by

$$W^s_\varepsilon(x) = \{ y \in X/d(g^n(x), g^n(y)) \leqslant \varepsilon, \forall n \geqslant 0\} \quad [\text{resp. } d(g^{-n}(x), g^{-n}(y)) \leqslant \varepsilon]$$

we have then

$$\forall y \in X, \quad d(g^n(x), g^n(y)) \leqslant \lambda^n \, d(x, y)$$

and therefore

$$W^s_\varepsilon(x) \subset \{ y \in X/d(g^n(x), g^n(y)) \to 0\} = W^s(x)$$

Proposition 1.1. g contracts in the stable direction and expands in the unstable direction.

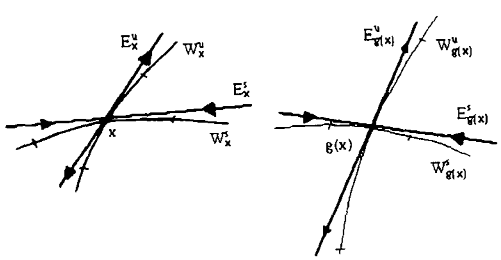
Definition 1.4. The "canonical coordinates"

$$\forall \delta > 0, \quad \exists \varepsilon > 0, \quad \forall (x, y) \in \Omega^2, \quad d(x, y) \leqslant \varepsilon \Rightarrow W^u_\delta(x) \cap W^s_\delta(y) = [x; y]$$

define a unique point and a continuous map (the local product)

$$[\cdot;\cdot]: \quad \{(x, y) \in \Omega^2/d(x, y) \leqslant \varepsilon\} \to \Omega$$

Proposition 1.2. $g|_\Omega$ is expansive (of constant γ).



Spectral Decomposition

We have $\Omega = \bigcup_{i=1}^k \Omega_i$, where the sets Ω_i are disjoint compact sets satisfying $g(\Omega_i) = \Omega_i$ and $g_{|\Omega_i}$ is topologically transitive.

Definition 1.5. The sets Ω_i are called *basic sets.*

Proposition 1.3. Any g-invariant measure has its support in Ω. In the particular case when g is an ergodic probability measure, its support is included in a basic set.

We introduce now Markov partitions to make an analogy with symbolic dynamical systems.

Definition 1.6. Let Λ be a basic set. A Markov partition is a finite cover $\mathcal{U} = (\mathcal{U}_i)_{i=1,\dots,m}$ of Λ made of proper rectangles ($R = \overline{\mathrm{int}(R)}$ and $\forall (x, y) \in R^2$, $[x; y] \in R$) such that

- $\mathrm{int}(\mathcal{U}_i) \cap \mathrm{int}(\mathcal{U}_j) = \varnothing$ for $i \neq j$.
- If $x \in \mathrm{int}(\mathcal{U}_i)$ and $g(x) \in \mathrm{int}(\mathcal{U}_j)$, then we have

$$W^u(g(x), \mathcal{U}_j) \subset g(W^u(x, \mathcal{U}_i)) \qquad \text{and} \qquad g(W^s(x, \mathcal{U}_i)) \subset W^s(g(x), \mathcal{U}_j)$$

with $W^s(x, \mathcal{U}_i) = W^s(x) \cap \mathcal{U}_i$.

We can make Markov partitions of arbitrary small diameter (in particular $< \gamma$). We associate to this partition the transition matrix $\mathbf{A}$ defined by

$$A_{ij} = \begin{cases} 1 & \text{if } \ \mathrm{int}(\mathcal{U}_i) \cap g^{-1}(\mathrm{int}(\mathcal{U}_j)) \neq \varnothing \\ 0 & \text{otherwise} \end{cases}$$

which is irreducible [$\forall (i, j)$, $\exists n$ such that $(A^n)_{ij} > 0$].

We define now the subshift of finite type associated to the matrix $\mathbf{A}$:

$$\Sigma_A = \{\underline{x} \in \{1,\dots, m\}^{\mathbb{Z}} / A_{x_n x_{n+1}} = 1\}$$
$$\Sigma_A^+ = \{\underline{x} \in \{1,\dots, m\}^{\mathbb{N}} / A_{x_n x_{n+1}} = 1\} \quad (\text{resp. } \Sigma_A^-)$$

On the compact set Σ_A we define a metric

$$\underline{d}(\underline{x}, \underline{y}) = \begin{cases} \lambda^k & \text{if } \ k = \sup\{|i| \colon x_i = y_i, \forall i, 0 \leqslant |i| < k\} \\ 0 & \text{if } \ \underline{x} = \underline{y} \end{cases}$$

and the shift σ: $\sigma(\underline{x}) = \underline{y}$, where $\forall n \geqslant 0$, $y_n = x_{n+1}$.

We can define a continuous surjection (Lipschitz)

$$\pi \colon \Sigma_A \to \Lambda$$
$$\underline{x} \to \bigcap_{j \in \mathbb{Z}} g^{-j}(\mathcal{U}_{x_j})$$

satisfying $\forall n \in \mathbb{Z}$, $\pi \circ \sigma^n = g^n \circ \pi$.

The map π represents a code of the orbits of points of Λ; moreover, π is bijective on $\Lambda \setminus \bigcup_{j \in \mathbb{Z}} g^{-j}(\partial^s \mathcal{U} \cup \partial^u \mathcal{U})$, where

$$\partial^s \mathcal{U} = \{x \in \mathcal{U} / x \notin \overline{\mathrm{int}(W^u(x, \mathcal{U})}\}$$

and $W^u(x, \mathcal{U}) = W^u_\varepsilon(x) \cap \mathcal{U}$ with $|\mathcal{U}| < \varepsilon$. The measure of the set $\bigcup_{j \in \mathbb{Z}} g^{-j}(\partial^s \mathcal{U} \cup \partial^u \mathcal{U})$ is 0 by $\|\cdot\|$ and by any Gibbs measure.[18]

Definition 1.7. $M(\Lambda)$ is the set of probability measures defined on Λ and $M_g(\Lambda)$ is the set of g-invariant probability measures defined on Λ.

Definition 1.8. $C(\Lambda)$ represents the set of real, continuous functions defined on Λ, and the set $C^\delta(\Lambda) \subset C(\Lambda)$ represents those which are δ-Hölder continuous.

Definition 1.9. The pressure of a function $\phi \in C^\delta(\Lambda)$ is the real

$$P_\phi = P_g(\phi) = \sup_{\rho \in M_g(\Lambda)} \left[h_\rho + \int \phi \, d\rho \right] \quad [= P_\sigma(\phi \circ \pi)]$$

and the unique measure μ_ϕ which achieves this supremum is the Gibbs measure of ϕ. To this measure μ_ϕ we associate the measure $\xi \in M_\sigma(\Sigma_A)$ such that $\mu_\phi = \pi^* \xi$. We have then $h_\xi(\sigma) = h_{\mu_\phi}(g)$ and the measure ξ is the Gibbs measure of $\phi \circ \pi \in C^\delta(\Sigma_A)$. The map $\pi: (\Lambda, \mu_\phi) \to (\Sigma_A, \xi)$ is an isomorphism of dynamical systems.[1, 18]

We decompose now the Bowen–Margulis measure μ (which is the Gibbs measure of 0) defined on the basic set $\Lambda \subset X$. Let h be the entropy of μ

$$h = \sup_{\rho \in M_g(\Lambda)} h_\rho > 0$$

We apply the Perron–Frobenius theorem to the matrices $\mathbf{A}$ and ${}^t\mathbf{A}$. There exist eigenvectors u and v and a real $\varphi(= e^h) > 1$ such that

$$Au = \varphi u \quad \text{and} \quad {}^tAv = \varphi v \quad \text{with} \quad \forall i \in [1; m], \quad u_i v_i > 0 \quad \text{and} \quad \sum_{i=1}^{m} u_i v_i = 1$$

We determine a measure ν as follows:

$$\text{for} \quad k \leqslant p, \qquad \nu\{\underline{x} \in \Sigma_A / x_k = y_k, ..., x_p = y_p\} = \varphi^{-(p-k)} v_{y_k} u_{y_p}$$

It is easy to see that (σ, ν) is a Markov chain over Σ_A which satisfies the relations

$$h_\nu(\sigma) = \operatorname{Log} \varphi = h = h_\mu(g)$$

and then we have

$$\mu = \pi^* \nu$$

We have, for example,

$$\forall i \in [1; m], \quad \nu\{\underline{x} \in \Sigma_A / x_0 = i\} = u_i v_i.$$

We define also ν^+ (resp. ν^-) on Σ_A^+ (resp. Σ_A^-) on the cylinders for $k \geqslant 0$ by

$$\begin{aligned} \nu^+\{\underline{x} \in \Sigma_A^+ / x_0 = y_0, \dots, x_k = y_k\} &= \varphi^{-k} u_{y_k} \\ [\text{resp.} \quad \nu^-\{\underline{x} \in \Sigma_A^- / x_{-k} = y_{-k}, \dots, x_0 = y_0\} &= \varphi^{-k} v_{y_{-k}}] \end{aligned} \tag{1.1}$$

We verify that

$$\sigma\nu^+_{|\{\underline{x} \in \Sigma_A^+ / x_0 = y_0,\, x_1 = y_1\}} = \varphi^{-1} \nu^+_{|\{\underline{x} \in \Sigma_A^+ / x_0 = y_1\}}$$

$$[\text{resp.} \quad \sigma^{-1}\nu^-_{|\{\underline{x} \in \Sigma_A^- / x_{-1} = y_{-1},\, x_0 = y_0\}} = \varphi^{-1} \nu^-_{|\{\underline{x} \in \Sigma_A^- / x_0 = y_{-1}\}}]$$

We associate to the measure ν^+(resp. ν^-) a measure μ^u (resp. μ^s) defined on the unstable manifolds W^u (resp. stable W^s). One proves that locally[19, 20]

$$\mu = \mu^u \times \mu^s \tag{1.2}$$

Consider now the dynamical partition (or Markovian partition) which is obtained by iterations of the Markov partition $\mathscr{U} = (\mathscr{U}_i)_{i=1,\dots,m}$ and defined by

$$\mathscr{P}_0 = \mathscr{U} \qquad \text{and} \qquad \mathscr{P}_n = \bigvee_{j=1-n}^{n-1} g^{-j}(\mathscr{P}_0) \tag{1.3}$$

Consider also the unstable dynamical partition $\mathscr{P}_n^u$ and the stable one $\mathscr{P}_n^s$ with

$$\mathscr{P}_n = [\mathscr{P}_n^u; \mathscr{P}_n^s] \tag{1.4}$$

We associate to an element U of $\mathscr{P}_n^u$ the element $y(U) \in U$ such that

$$|g^n(U)| = |(g^n)'(y(U))| \cdot |U| \simeq 1 \tag{1.5}$$

Here and throughout this paper, the sign $\simeq$ expresses that the ratios of both sides are uniformly bounded by constants c and c^{-1}. We have similar properties for elements of $\mathscr{P}_n^s$, and we have, following (1.1), for $U \in \mathscr{P}_n^u$ and $V \in \mathscr{P}_n^s$,

$$\mu^u(U) \simeq \mu^s(V) \simeq e^{-nh} \tag{1.6}$$

since U and V are associated under π with cylinders of size n in, respectively, Σ_A^+ and Σ_A^- and centered in, respectively, $\pi^{-1}(y(U))$ and $\pi^{-1}(y(V))$. It is easy to see that there exist two positive reals $1 < a < b$ such that for any $U \in \Sigma_A^+$ and $V \in \Sigma_A^-$

$$b^{-n} \leqslant |U|, |V| \leqslant a^{-n} \tag{1.7}$$

Let us note for $x \in \Lambda$

$$J^u(x) = -\text{Log Jacobian } Dg\colon E^u_x \to E^u_{g(x)}$$
$$[\text{resp.}\quad J^s(x) = \text{Log Jacobian } Dg\colon E^s_x \to E^s_{g(x)}] \tag{1.8}$$

The functions J^u and J^s are negative and δ-Hölder continuous functions.[1] Expression (1.5) becomes

$$\exp\left\{\sum_{j=0}^{n-1} J^u[g^j(y(U))]\right\} \simeq |U|$$
$$\left(\text{resp.}\quad \exp\left\{\sum_{j=0}^{n-1} J^s[g^j(y(V))]\right\} \simeq |V|\right) \tag{1.9}$$

We shall use (1.2) to decompose the free energy function F into $F^u + F^s$ where F^u (resp. F^s) is an unstable free energy function (resp. stable). Following ref. 2, if we take f as in (0.4), we prove that f is the dimension spectrum of the measure μ. This function $f(\alpha)$ is also related to the Hausdorff dimensions of measures μ_α whose supports are the singularity sets C_α. We decompose the sets C_α into the local product of singularity sets of μ^u and μ^s: $C_\alpha \supset [C_{\alpha^u}; C_{\alpha^s}]$, and these two sets have the same Hausdorff dimensions.

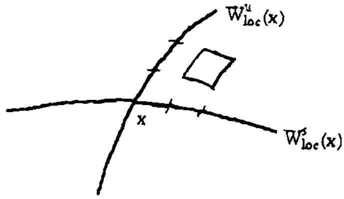
The first step is to introduce the free energy function and to compute it.

2. EXISTENCE AND REGULARITY OF THE FREE ENERGY FUNCTION

We shall see that the existence is very much harder to prove than the regularity. For the existence we resolve this two-dimensional problem into a one-dimensional problem by decomposing the free energy function F into the sum of an unstable free energy function F^u and a stable free energy function F^s. This is done in the next section.

2.1. Decomposition of the Free Energy Function

Each unstable manifold intersects Λ in a countable union of sets of type $W^u_{\text{loc}}(x)$, and intersects transversally all the stable manifolds. The metrics on $W^u_{\text{loc}}(x) \cap \Lambda$, which is compact, as a Riemannian submanifold, and the one inducted as the restriction of $\|\cdot\|$ are Lipschitz equivalent. Then the choice of the metric does not matter in the following. We shall use, for example, uniform partitions $(U^u_n)_{n \geqslant 1}$ [resp. $(U^s_n)_{n \geqslant 1}$] on the unstable (resp. stable) manifolds of diameter $\simeq e^{-n}$. We define then the local product $U_n = [U^u_n; U^s_n]$.



We define also the real functions as in (0.5) for any real β

$$F^u_n(\beta) = -\frac{1}{n}\,\mathrm{Log}\left\{\sum_{U \in U^u_n} \mu^u(U)^\beta\right\} \quad [\text{resp. } F^s_n(\beta) \text{ for } V \in U^s_n] \quad (2.1.1)$$

We have at the rank n with $W = [U, V]$

$$F_n(\beta) = -\frac{1}{n}\,\mathrm{Log}\left\{\sum_{W \in U_n} \mu(W)^\beta\right\} \tag{2.1.2}$$

Using (1.2), we have locally

$$\mu(W) = \mu^u(U)\,\mu^s(V)$$

and (2.1.2) becomes

$$F_n(\beta) = F^u_n(\beta) + F^s_n(\beta) + u_n(\beta) \tag{2.1.3}$$

where $u_n(\beta)$ represents parasite terms which disappear at the limit (with $1/n$ Log) when n goes to $+\infty$. It suffices therefore to show that the sequence of functions $(F^u_n)_{n \geqslant 1}$ [resp. $(F^s_n)_{n \geqslant 1}$] converges to a function F^u (resp. F^s) to obtain

$$\forall \beta \in \mathbb{R}, \quad \lim_{n \to +\infty} F_n(\beta) = F(\beta) = F^u(\beta) + F^s(\beta) \tag{2.1.4}$$

We compute then in the next section the functions F^u and F^s in order to obtain F.

2.2. Computation of the Free Energy Function

The unstable free energy function is given by the following.

Theorem 2.2.1. We have for any real β

$$F^u(\beta) = \inf_{\rho \in M_g(\Lambda)} \left[\frac{h_\rho - \beta h}{\int J^u\, d\rho}\right]$$

Observe that there is nothing to prove for $\beta = 1$, since we have for any partition $(U^u_n)_{n \geqslant 1}$, $F^u_n(1) = 0$, and then both quantities are 0. Observe also that the functional involved in the theorem satisfies the following result (or its opposite if we take $-F^u$).

Proposition 2.2.2. We have for any real β

$$\sup_{\rho \in M_g(\Lambda)} I(\rho) = \sup_{\substack{\rho \in M_g(\Lambda) \\ \rho \text{ ergodic}}} I(\rho)$$

where

$$I(\rho) = \frac{h_\rho - \beta h}{\int -J^u \, d\rho}$$

Proof of Proposition 2.2.2. The map $\rho \to I(\rho)$ is upper semicontinuous since the dynamical system expands (ref. 4, 16.7, p. 107). The ergodic measures are extremal and form a G_δ in $M_g(\Lambda)$—this property comes from the specification in ref. 4, 21.9, p. 198. The supremum is then equal over the two sets, and it is achieved since $M_g(\Lambda)$ is compact. ∎

Remarks. The functional I is a large-deviations functional.

- As proved with regard to Theorem 2.4.1, this supremum is achieved by a unique measure μ_β^u which is the Gibbs measure of the Hölder continuous function $-h\beta - F^u(\beta) J^u$.
- We have for $\beta > 1$, $\forall \xi \in M_g(\Lambda)$, $I(\xi) < 0$, and for $\beta = 1$, $\forall \xi \neq \mu_1^u$, $I(\xi) < 0$.

Theorem 2.2.1 will follow directly from Lemmas 2.2.3 and 2.2.5. We first estimate an upper bound of the upper limit with the following.

Lemma 2.2.3. We have for any real β

$$\overline{\lim_{n \to +\infty}} \; -F_n^u(\beta) \leqslant \sup_{\rho \in M_g(\Lambda)} I(\rho)$$

Proof of Lemma 2.2.3. Let $\beta \in \mathbb{R}$ and $(U_n^u)_{n \geqslant 1}$ be a uniform unstable partition such that for any $U \in U_n^u$ we have $|U| \simeq e^{-n}$. Using (1.6) and (1.9), we associate to any interval $U \in U_n^u$ an integer $n(U)$ and an element $y(U) \in U$ such that

$$|g^{n(U)}(U)| = |U| \exp\left\{\sum_{j=0}^{n(U)-1} J^u[g^j(y(U))]\right\} \simeq 1$$

or in another form

$$\sum_{j=0}^{n(U)-1} J^u[g^j(y(U))] \sim -n \tag{2.2.1}$$

and the μ^u measure of U satisfies

$$\mu^u(U) \simeq e^{-n(U)h} \tag{2.2.2}$$

It suffices to see that if

$$y(U) = \pi((y_i)_{i \geqslant 1}) \quad \text{and} \quad C_k^+ = \{\underline{x} \in \Sigma_A^+ / x_i = y_i, 0 \leqslant i \leqslant k\} \text{ for } k > 0$$

and $p = \inf\{k / \exists C_k^+ \subset \pi^{-1}(U)\} \geqslant n(U)$, then we have

$$\mu^u(g^{n(U)}(U)) \simeq 1 \simeq \nu^+(\sigma^p(C_p^+)) \quad \text{and} \quad \mu^u(U) \simeq e^{-n(U)h} \simeq \nu^+(C_p^+) \simeq e^{-ph}$$

[$n(U)$ represents the "size" of U and e^{-n} its length]. We have therefore by (2.2.2)

$$-F_n^u(\beta) = \frac{1}{n} \log\left\{ \sum_{U \in U_n^u} \mu^u(U)^\beta \right\}$$

and this leads to

$$-F_n^u(\beta) \sim \frac{1}{n} \operatorname{Log}\left\{ \sum_{U \in U_n^u} e^{-n(U)\beta h} \right\} \tag{2.2.3}$$

Let us define the sets

$$E_i = \{ U \in U_n^u / n(U) = i \} \tag{2.2.4}$$

which are only defined for integers i varying in a linear scale, since, using (2.2.1),

$$i \in \left[\frac{n}{\sup -J^u}; \frac{n}{\inf -J^u} \right] = [na_1; na_2]$$

There exists therefore an integer $i(n)$ such that for any integer i we have

$$\# E_i e^{-i\beta h} \leqslant \# E_{i(n)} e^{-i(n)\beta h}$$

and then

$$\# E_{i(n)} e^{-i(n)\beta h} \leqslant \sum_{U \in U_n^u} e^{-n(U)\beta h} \leqslant (a_2 - a_1) n \ \# \ E_{i(n)} e^{-i(n)\beta h}$$

and (2.2.3) becomes

$$-F_n^u(\beta) \sim \frac{1}{n} \operatorname{Log} \# E_{i(n)} e^{-i(n)\beta h}$$

or

$$-F_n^u(\beta) \sim \frac{1}{n} \operatorname{Log} \# E_i(n) - \frac{i(n)}{n} \beta h \tag{2.2.5}$$

Let us define the probability measures

$$\theta_n = \frac{1}{\# E_{i(n)}} \sum_{U \in E_{i(n)}} \delta_{y(U)} \qquad \text{and} \qquad \xi_n = \frac{1}{i(n)} \sum_{j=0}^{i(n)-1} g^j \theta_n$$

The sequences

$$\frac{1}{n} \operatorname{Log} \# E_{i(n)} \in [0; 1], \qquad \frac{i(n)}{n} \in [a_1; a_2], \qquad \xi_n \in M(\Lambda)$$

take their values in compact sets. We can suppose, if necessary by reindexing the sequences, that these sequences converge:

$$\begin{gathered} \frac{1}{n} \operatorname{Log} \# E_{i(n)} \to \gamma \in [0; 1] \\ \frac{i(n)}{n} \to \eta \in [a_1; a_2] \\ \xi_n \to \xi \qquad \text{(observe that the limit is } g\text{-invariant)} \end{gathered} \tag{2.2.6}$$

We get therefore

$$-F_n^u(\beta) \to \gamma - \eta \beta h \tag{2.2.7}$$

Let us compute the integral

$$\int J^u \, d\xi_n = \frac{1}{\# E_{i(n)}} \sum_{U \in E_{i(n)}} \left\{ \frac{1}{i(n)} \sum_{j=0}^{i(n)-1} J^u[g^j(y(U))] \right\}$$

Using (2.2.1) and (2.2.4), we have for any $U \in E_{i(n)}$

$$\frac{1}{i(n)} \sum_{j=0}^{i(n)-1} J^u[g^j(y(U))] \sim \frac{-n}{i(n)}$$

and then

$$\int J^u \, d\xi_n \sim \frac{-n}{i(n)}$$

We get therefore

$$\lim_{n \to +\infty} \frac{i(n)}{n} = \eta = \frac{1}{\int -J^u \, d\xi} \tag{2.2.8}$$

Now we estimate the number γ by a standard argument due to Misiurewicz (ref. 4, p. 145) with the following result.

Proposition 2.2.4. We have

$$\gamma \leqslant \frac{h_\xi}{\int -J^u \, d\xi}$$

Proof of Proposition 2.2.4. Let (P) be a ξ continuous partition (for example, the unstable dynamical partition) whose elements have diameter $\delta < \lambda$. To obtain a lower estimate of the ξ-entropy of the partition (P), we define the iterations

$$(P)^j = \bigvee_{k=0}^{j} g^{-k}(P)$$

and we compute for any integer M the number $H_{\xi_n}(P)^M$. Let us recall that each set E_i is (i, δ)-separated[11] $(\forall x \neq y), \exists j < i, d(g^j(x), g^j(y)) > \delta)$, since the associated cylinder satisfies $\underline{d}(\sigma^j(\underline{x}), \sigma^j(\underline{y})) > \delta$. The set $E_{i(n)}$ is therefore $(i(n), \delta)$-separated, and we have for any $B \in (P)^{i(n)-1}$, $\#\{B \cap E_{i(n)}\} \leqslant 1$. The classical computation of the entropy follows:

$$\frac{1}{i(n)} H_{\theta_n}(P)^{i(n)-1} = \frac{-1}{i(n)} \sum_{U \in E_{i(n)}} \frac{1}{\# E_{i(n)}} \operatorname{Log} \frac{1}{\# E_{i(n)}} = \frac{1}{i(n)} \operatorname{Log} \# E_{i(n)}$$

and taking the limit when n goes to $+\infty$, we get with (2.2.6)

$$\lim_{n \to +\infty} \frac{1}{i(n)} H_{\theta_n}(P)^{i(n)-1} = \frac{\gamma}{\eta} \tag{2.2.9}$$

For an integer M such that $i(n) > 2M$, we define for integers $q \in [0; M-1]$

$$s(q) = \operatorname{Int}\left(\frac{i(n) - q}{M}\right) \qquad \text{and} \qquad R_q = \{0, 1, \ldots, q-1, s(q)\, M + q, \ldots, i(n) - 1\}$$

and we have $\# R_q \leqslant 2M$. We obtain for $q \in [0; M-1]$

$$(P)^{i(n)-1} = \bigvee_{k=0}^{s(q)-1} g^{-kM-q}(P)^{M-1} \vee \bigvee_{m \in R_q} g^{-m}(P)$$

and we have then

$$H_{\theta_n}(P)^{i(n)-1} \leqslant \sum_{k=0}^{s(q)-1} H_{\theta_n}[g^{-kM-q}(P)^{M-1}] + \# R_q \operatorname{Log} \# P$$

We verify that

$$\sum_{j=0}^{i(n)-1} H_{g^j\theta_n}(P)^{M-1} = \sum_{j=0}^{i(n)-1} H_{\theta_n}[g^{-j}(P)^{M-1}]$$
$$\geqslant \sum_{q=0}^{M-1} \sum_{k=0}^{s(q)-1} H_{\theta_n}[g^{-kM-q}(P)^{M-1}] \qquad (*)$$

$$\sum_{k=0}^{s(q)-1} H_{\theta_n}[g^{-kM-q}(P)^{M-1}] \geqslant H_{\theta_n}(P)^{i(n)-1} - \# R_q \operatorname{Log} \# P$$
$$\geqslant H_{\theta_n}(P)^{i(n)-1} - 2M \operatorname{Log} \# P \qquad (**)$$

The expressions $(*)$ and $(**)$ lead to

$$\sum_{j=0}^{i(n)-1} H_{g^j\theta_n}(P)^{M-1} \geqslant MH_{\theta_n}(P)^{i(n)-1} - 2M^2 \operatorname{Log} \# P \qquad (***)$$

Using the concavity of the function $x \to -x \operatorname{Log} x$, we get

$$H_{\xi_n}(P)^{M-1} \geqslant \frac{1}{i(n)} \sum_{j=0}^{i(n)-1} H_{g^j\theta_n}(P)^{M-1} \qquad (****)$$

and we obtain by comparing $(***)$ and $(****)$

$$H_{\xi_n}(P)^{M-1} \geqslant \frac{M}{i(n)} H_{\theta_n}(P)^{i(n)-1} - \frac{2M^2}{i(n)} \operatorname{Log} \# P \qquad (2.2.10)$$

Taking the limit in (2.2.10) when n goes to $+\infty$, we obtain with (2.2.9)

$$\frac{1}{M} H_{\xi}(P)^{M-1} \geqslant \frac{\gamma}{\eta} \qquad (2.2.11)$$

Taking the limit in (2.2.11) when M goes to $+\infty$, we get

$$h_{\xi} \geqslant \frac{\gamma}{\eta}$$

and comparing to (2.2.8) achieves Proposition 2.2.4. ∎

Using Proposition 2.2.4 and (2.2.7), we get

$$-F_n''(\beta) \to \gamma - \eta\beta h \leqslant I(\xi)$$

We have then proved a stronger result, which says that for any cluster point F of the sequence $(-F_n^u(\beta))_{n \geqslant 1}$ there exists a g-invariant measure ξ which satisfies the inequality $F \leqslant I(\xi)$, and that gives obviously Lemma 2.2.3. ∎

We prove now a sort of reverse inequality, since we estimate a lower bound of the lower limit with the following result.

Lemma 2.2.5. We have for any real β

$$\varliminf_{n \to +\infty} -F_n^u(\beta) \geqslant \sup_{\substack{\rho \in M_g(A) \\ \rho \text{ ergodic}}} I(\rho)$$

Proof of Lemma 2.2.5. We consider a g-invariant ergodic measure ρ. From the ergodic theorem we have on a set of ρ measure 1

$$\lim_{n \to +\infty} \frac{1}{i(n)} \sum_{j=0}^{i(n)-1} J^u[g^j(y)] = \int J^u \, d\rho \tag{2.2.12}$$

Using (2.2.1) and (2.2.12), the theorem of Shannon–McMillan (ref. 4, p. 81), leads us to consider the intervals U of U_n^u such that

$$n(U) \sim \frac{n}{\int -J^u \, d\rho} = n_\rho \tag{2.2.13}$$

since the measure ρ is concentrated on these elements. Let $\varepsilon > 0$ and the set

$$A_\rho^n = \{ U \in U_n^u / n_\rho - \varepsilon < n(U) < n_\rho + \varepsilon \}$$

There exists an integer N such that for any integer $n \geqslant N$ we get $\rho(A_\rho^n) \geqslant 1 - \varepsilon$ and

$$\# A_\rho^n \geqslant (1 - \varepsilon) \exp\{ n_\rho (h_\rho - \varepsilon) \} \tag{2.2.14}$$

and for any interval $U \in A_\rho^n$ we have

$$\mu^u(U)^\beta \geqslant \exp\{ -n_\rho (h_\rho \pm \varepsilon) \, h\beta \}. \tag{2.2.15}$$

From the definition (2.1.1) and the inequalities (2.2.14) and (2.2.15) we get

$$-F_n^u(\beta) \geqslant \frac{1}{n} \operatorname{Log} \left\{ \sum_{U \in A_\rho^n} \mu^u(U)^\beta \right\} \geqslant \frac{1}{n} \operatorname{Log} \# A_\rho^n \exp\{ -n_\rho (h_\rho \pm \varepsilon) \, h\beta \}$$

and this leads to the inequality

$$\varliminf_{n \to +\infty} -F_n^u(\beta) \geqslant \frac{h_\rho - \beta h - 2\varepsilon}{\int -J^u \, d\rho + \varepsilon}$$

Since the real ε is arbitrary we get

$$\varliminf_{n\to+\infty} -F_n^u(\beta) \geqslant I(\rho)$$

and since the measure ρ is arbitrary we get Lemma 2.2.5. ∎

Using Proposition 2.2.2 and Lemma 2.2.5, we obtain finally

$$\varliminf_{n\to+\infty} -F_n^u(\beta) \geqslant \sup_{\rho\in M_g(A)} I(\rho)$$

which, with Lemma 2.2.3, gives Theorem 2.2.1. ∎

We prove the following similar result for the stable free energy function.

Theorem 2.2.6. We have for any real β

$$F^s(\beta) = \inf_{\rho\in M_g(A)} \left[\frac{h_\rho - \beta h}{\int J^s\, d\rho}\right]$$

The proof is analogous to the one of Theorem 2.2.1; it suffices to take g^{-1} instead of g ("unstable under g^{-1} becomes stable under g"). We get then

$$J^u_{g^{-1}} = J^s_g, \qquad W^u_{\mathrm{loc}}(x, g^{-1}) = W^s_{\mathrm{loc}}(x, g) \qquad \text{and} \qquad U^u_{n, g^{-1}} = U^s_{n, g}$$

Proposition 2.2.7. The functions F^u, F^s, and $F = F^u + F^s$ are concave.

We are interested now in a more intrinsic free energy function (the dynamical one) which is generated by the dynamical partition.

2.3. Computation of the Dynamical Free Energy Function

We define on $\mathbb{R}^2$ the unstable (resp. stable) dynamical free energy function by

$$\forall (x, y)\in\mathbb{R}^2, \quad G_n^u(x, y) = \frac{1}{n}\operatorname{Log}\left\{\sum_{A\in\mathscr{P}_n^u} \mu^u(A)^x\,|A|^y\right\} \quad [\text{resp. } G_n^s(x, y)]$$

and the convergence of these sequences is treated in the following theorem.

Theorem 2.3.1. We have for any pair $(x, y)\in\mathbb{R}^2$

$$\lim_{n\to+\infty} G_n^u(x, y) = G^u(x, y) = \sup_{\rho\in M_g(A)} \left[h_\rho + \int (yJ^u - hx)\, d\rho\right]$$

$$\lim_{n\to+\infty} G_n^s(x, y) = G^s(x, y) = \sup_{\rho\in M_g(A)} \left[h_\rho + \int (yJ^s - hx)\, d\rho\right]$$

The proof of Theorem 2.3.1 parallels the proof of Theorems 2.2.1 and 2.2.6; the major fact is that for any $A \in \mathscr{P}_n^u$ and $B \in \mathscr{P}_n^s$ we have $\mu^u(A) \simeq \mu^s(B) \simeq e^{-nh}$ and

$$\exp\left\{\sum_{j=0}^{n-1} J^u[g^j(y(A))]\right\} \simeq |A| \qquad \text{and} \qquad \exp\left\{\sum_{j=0}^{n-1} J^s[g^j(y(B))]\right\} \simeq |B|$$

So it is easy to prove that, for example,

$$\overline{\lim_{n\to+\infty}}\; G_n^u(x, y) \leqslant \sup_{\rho \in M_g(A)} \left[h_\rho + \int (yJ^u - hx)\, d\rho\right]$$

An analogous counting argument to the one in Lemma 2.2.5 shows that

$$\underline{\lim_{n\to+\infty}}\; G_n^u(x, y) \geqslant \sup_{\substack{\rho \in M_g(A)\\ \rho \text{ ergodic}}} \left[h_\rho + \int (yJ^u - hx)\, d\rho\right]$$

Since this functional is convex (and upper semicontinuous) and the ergodic measures are extremal (and form a G_δ), we have

$$\sup_{\substack{\rho \in M_g(A)\\ \rho \text{ ergodic}}} \left[h_\rho + \int (yJ^u - hx)\, d\rho\right] = \sup_{\rho \in M_g(A)} \left[h_\rho + \int (yJ^u - hx)\, d\rho\right]$$

Observe that $G^u(x, y)$ [resp. $G^s(x, y)$] represents the pressure of the Hölder continuous function $yJ^u - hx$ (resp. $yJ^s - hx$) and then the supremum is achieved by a unique measure which is its Gibbs measure. Moreover, these functions G^u and G^s are real analytic in both variables x and y.[18] This property will help us to prove the smoothness of the free energy function F.

2.4. Regularity of the Free Energy Function F

The relation between F^u and G^u, and F^s and G^s is derived from the following.

Theorem 2.4.1. We have for any real β

$$G^u(\beta, -F^u(\beta)) = G^s(\beta, -F^s(\beta)) = 0$$

Proof of Theorem 2.4.1. Let $\beta \in \mathbb{R}$. We have from Theorem 2.2.1 for any g-invariant measure ξ

$$h_\xi - h\beta - F^u(\beta) \int J^u\, d\xi \leqslant 0 \tag{2.4.1}$$

This leads us to the inequality

$$G^u(\beta, -F^u(\beta)) \leqslant 0$$

Since the function $\tau_\beta = F^u(\beta) J^u - h\beta$ is Hölder continuous, the pressure (or G^u) is achieved by a unique measure μ_β^u which is also the unique measure which achieves $F^u(\beta)$. The inequality (2.4.1) becomes an equality only for $\xi = \mu_\beta^u$ and we have

$$G^u(\beta, -F^u(\beta)) = P(\tau_\beta) = 0 \tag{2.4.2}$$

By the same method we prove that $G^s(\beta, -F^s(\beta))$ and $F^s(\beta)$ are achieved by a unique measure μ_β^s which is the Gibbs measure of the Hölder continuous function $\eta_\beta = F^s(\beta) J^s - h\beta$, and this gives Theorem 2.4.1. ∎

The smoothness of the function F^u is given by the following result.

Theorem 2.4.2. The function F^u is real analytic on $\mathbb{R}$, is strictly increasing, and is either linear (this is the case when J^u is homologous to a constant, i.e., $J^u = C + K \circ g - K$) or strictly concave.

Proof of Theorem 2.4.2. We differentiate (2.4.2) and we obtain for any real β

$$\left[\left(\frac{\partial G^u}{\partial x}\right) - (F^u)'(\beta)\left(\frac{\partial G^u}{\partial y}\right)\right](\beta, -F^u(\beta)) = 0$$

We get from[13, 14, 18]

$$\left(\frac{\partial G^u}{\partial y}\right)(\beta, -F^u(\beta)) = \int J^u \, d\mu_\beta^u$$

and then we obtain

$$(F^u)'(\beta) = \frac{h}{\int -J^u \, d\mu_\beta^u} > 0 \tag{2.4.3}$$

From a theorem of implicit functions and since $(\partial G^u/\partial y)(\beta, -F^u(\beta)) < 0$ (and $\neq 0$), the function F^u is real analytic. Differentiating one more time (2.4.2), we obtain

$$\begin{aligned}(F^u)''(\beta) &= h^2 \left[\frac{(\partial^2 G^u/\partial y^2)}{(\partial G^u/\partial y)^3}\right](\beta, -F^u(\beta)) \\ &= \frac{h^2}{(\int J^u \, d\mu_\beta^u)^3}\left(\frac{\partial^2 G^u}{\partial y^2}\right)(\beta, -F^u(\beta))\end{aligned} \tag{2.4.4}$$

We have from[14, 15]

$$\left(\frac{\partial^2 G^u}{\partial y^2}\right) > 0$$

except for the case when $J^u = C + K \circ g - K$, which is the degenerate case.[18] We have then in the general case

$$\forall \beta \in \mathbb{R}, \quad (F^u)''(\beta) < 0$$

and this gives Theorem 2.4.2 ∎

We obviously have similar results for F^s. We obtain then the following result.

Proposition 2.4.3. The function F is real analytic on $\mathbb{R}$, is strictly increasing, and is either linear (this is the case when μ is absolutely continuous with respect to $|\cdot|$) or strictly concave.

The case when F is degenerate is the case when J^u and J^s are homologous to constants (F^u and F^s are degenerate); the function J, the logarithm of the Jacobian of g ($= J^s - J^u$), is also homologous to a constant: $J = C + K \circ g - K$. The measure μ is then absolutely continuous with respect to the Lebesgue measure, like the Gibbs measures μ_{J^u} and μ_{J^s} of the type $ke^{K(x)} |\cdot|$.

Let us define the Legendre–Fenchel transforms of F^u, F^s, and F from (0.4): f^u, f^s, and f. In the case when F is linear (resp. F^u and F^s) the function f (resp. f^u and f^s) is only defined in one point:

$$F(\beta) = \frac{-1}{c} h(1-\beta) \qquad \text{and} \qquad f\left(\frac{h}{c}\right) = \frac{h}{c}$$

which is the degenerate case. In the general case f is defined and positive on an interval $]\alpha_1 ; \alpha_2[\subset \mathbb{R}^{+*}$ (possibilities of limits at the boundaries) and $f \equiv -\infty$ otherwise ($\alpha < \alpha_1$ and $\alpha > \alpha_2$); f is also concave, and by a relation of conjugacy we have[5]

$$f(\alpha) + F(\beta) = \alpha\beta \Leftrightarrow \alpha = F'(\beta) \tag{2.4.5}$$

We have then for any real β

$$f(F'(\beta)) = \beta F'(\beta) - F(\beta) \tag{2.4.6}$$

which means that f is real analytic on $]\alpha_1 ; \alpha_2[$. The expression (2.4.5) leads to

$$f(\alpha) + F(\beta) = \alpha\beta \Leftrightarrow \beta = f'(\alpha) \tag{2.4.7}$$

Let us define $\alpha(0) = F'(0)$. We have then:

- f is strictly increasing on $]\alpha_1; \alpha(0)[$.
- f achieves its supremum at $\alpha(0)$.
- f is strictly decreasing on $]\alpha(0); \alpha_2[$.
- For $\alpha \in]\alpha_1; \alpha_2[$ and $\beta = f'(\alpha)$ we have

$$f''(\alpha) = \frac{1}{F''(\beta)}$$

and f is strictly concave.

We have of course similar results for f^u and f^s. In the next section we relate the function f to the dimension spectrum of the measure μ, i.e., to the Hausdorff dimensions of the singularity sets of the measure μ.

3. COMPUTATION OF THE HAUSDORFF DIMENSIONS OF THE SINGULARITY SETS OF THE MEASURE μ

Let D_α be the set of points x such that α is a cluster point of

$$\frac{\text{Log}\,\mu(R)}{\text{Log}\,|R|} \quad \text{where} \quad x \in \text{int}(R) \quad \text{and} \quad |R| \to 0$$

We have then the following result.

Theorem 3.1. We have for any real $\alpha \in]\alpha_1; \alpha_2[$

$$HD(D_\alpha) = HD(C_\alpha^+) = HD(C_\alpha^-) = HD(C_\alpha) = f(\alpha)$$

Proof of Theorem 3.1. We define the product partition $U_n = [U_n^u; U_n^s]$ and the random variables $(W_n)_{n \geqslant 1}$ on U_n by $W_n = \text{Log}\,\mu(U)$ and equipped with the counting probability measure. Using (0.5), we have for any real β

$$\mathbb{E}(\exp(\beta W_n)) = \frac{Z_n(\beta)}{e^n}$$

and with Theorem 2.2.1 we obtain obviously

$$\lim_{n \to +\infty} \frac{1}{n} \text{Log}\{\mathbb{E}(\exp(\beta W_n))\} = -F(\beta) - 1$$

The sequence $(W_n)_{n \geqslant 1}$ satisfies a large-deviations theorem (ref. 5, II6.1). For any real $x \in D_\alpha$ there exists a sequence $(R_p)_{p \geqslant 1}$ such that

$$x \in \text{int}(R_p), \qquad |R_p| \to 0 \qquad \text{and} \qquad \lim_{p \to +\infty} \frac{\text{Log}\,\mu(R_p)}{\text{Log}\,|R_p|} = \alpha$$

For $\varepsilon > 0$ (such that $\alpha + \varepsilon \in]\alpha_1; \alpha_2[$) and according to the large-deviations theorem, there exists an integer N such that for any integer $n > N$ we have

$$\#\{U \in U_n^u / \mu(U) \geqslant |U|^{\alpha+\varepsilon}\} \leqslant \exp\{n(f(\alpha+\varepsilon)+\varepsilon)\} \tag{3.1}$$

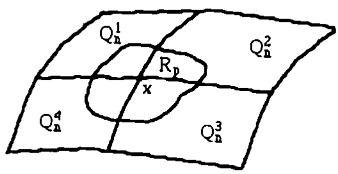
Let $m \geqslant N$ an integer and $\delta = e^{-m}$ a positive real; for large integers p we get

$$|R_p| \leqslant \delta c \qquad \text{and} \qquad \mu(R_p) \geqslant |R_p|^{\alpha+\varepsilon/2}$$

and there exists an integer $n(p) > m$ such that

$$\frac{1}{c} e^{-n} \leqslant |R_p| \leqslant c e^{-n}$$

We describe the general situation as follows:



We have then

$$\mu(R_p) \leqslant \mu\left(\bigcup_i Q_n^i\right) \leqslant 4 \sup_i \mu(Q_n^i)$$

and also

$$\sup_i \mu(Q_n^i) \geqslant 4^{-1}\mu(R_p) \geqslant 4^{-1} |R_p|^{\alpha+\varepsilon/2} \geqslant c^{-\alpha-\varepsilon/2} 4^{-1} |Q_n^i|^{\alpha+\varepsilon/2} \geqslant |Q_n^i|^{\alpha+\varepsilon}$$

We have then the property

$$\exists i, \quad 1 \leqslant i \leqslant 4, \quad \mu(Q_n^i) \geqslant |Q_n^i|^{\alpha+\varepsilon}$$

The element R_p is contained in a ball of diameter ce^{-n} and centered in Q_n^i. Let the real $\tau > f(\alpha+\varepsilon)+\varepsilon$. We get with the definition of the Hausdorff dimension

$$HDM_{\tau,\delta}(D_\alpha) = \inf_{\substack{D_\alpha \subset \bigcup_i P_i \\ |P_i| \leqslant \delta}} \sum_i |P_i|^\tau \leqslant \sum_{n \geqslant m} \exp\{n(f(\alpha+\varepsilon)+\varepsilon)\}(ce^{-n})^\tau$$

since the number $\exp\{n(f(\alpha+\varepsilon)+\varepsilon)\}$ is greater than the number of elements $U \in U_n^u$ satisfying $\mu(U) \geqslant |U|^{\alpha+\varepsilon}$. We find

$$HDM_{\tau,\delta}(D_\alpha) \leqslant c^\tau \sum_{n \geqslant m} \exp\{n(f(\alpha+\varepsilon)+\varepsilon-\tau)\} \tag{3.2}$$

which is a convergent series by our assumption. Since δ goes to 0 when m goes to $+\infty$, we get

$$\lim_{\delta \to 0} HDM_{\tau,\delta}(D_\alpha) = 0 \tag{3.3}$$

This leads to the inequality

$$HD(D_\alpha) \leqslant f(\alpha + \varepsilon) + \varepsilon$$

and since ε is arbitrary, we obtain

$$HD(D_\alpha) \leqslant f(\alpha) \tag{3.4}$$

We use Frostman's lemma applied to an appropriate measure to prove the reverse inequality. Let us recall that in ref. 2 one constructs recursively a familly $(R_p)_{p \geqslant 1}$ of intervals depending on two sequences $(r_j)_{j \geqslant 1}$ and $(\delta_j)_{j \geqslant 1}$. The assumptions made on these two sequences allow us to define two families of intervals $(R_p^u)_{p \geqslant 1}$ and $(R_p^s)_{p \geqslant 1}$. Let us define the sequence $(R_p)_{p \geqslant 1}$:

$$R_p = [R_p^u; R_p^s]$$

and the set

$$V_\alpha = \bigcap_{p \geqslant 1} \left(\bigcup_{R \in R_p} R \right)$$

The first step is to prove that

$$V_\alpha \subset C_\alpha = (C_\alpha^+ \cap C_\alpha^-) \subset D_\alpha.$$

Then we define recursively a measure ξ on R_p satisfying $\xi(R_1) = 1$ and for any $R \in R_p$

$$\xi(R) = \frac{\xi(R')}{\#\{H \in R_p / H \subset R'\}}$$

where R' is the only element of R_{p-1} such that $R \subset R'$ [$\xi(\cdot \mid R')$ is the counting probability measure on R_p]. The measure ξ is the product of the measures ξ^u and ξ^s obtained by the same method on the unstable and stable manifolds and which satisfy Frostman's lemma. The result can be applied to ξ since we have

$$\xi(B_\sigma) \leqslant C\, |B_\sigma|^{f(\alpha)}$$

where β_σ is a ball of small diameter σ. We obtain therefore the inequality

$$HD(V_\alpha) \geqslant f(\alpha) \tag{3.5}$$

Since we have

$$V_\alpha \subset C_\alpha = (C_\alpha^+ \cap C_\alpha^-) \subset (C_\alpha^+ \cup C_\alpha^-) \subset D_\alpha$$

we get following (3.4) and (3.5) a stronger result

$$HD(V_\alpha) = HD(C_\alpha) = HD(C_\alpha^\pm) = HD(D_\alpha) = f(\alpha)$$

and this gives Theorem 3.1. ▮

In the next section we study more precisely the function f, in particular at the boundaries α_1 and α_2.

4. PROOF OF THE DIMENSION SPECTRUM THEOREM

In the general case we obtain results which generalize those in dimension one[17] and are given by the following.

Theorem 4.1. For any real $\alpha \in [\alpha_1; \alpha_2]$ there exists a g-invariant measure μ_α such that

$$f(\alpha) = HD(\mu_\alpha) \qquad \text{and} \qquad \frac{\operatorname{Log} \mu(R)}{\operatorname{Log} |R|} \xrightarrow[|R| \to 0]{} \alpha \quad \mu_\alpha \quad \text{a.e.}$$

Moreover, there exist positive reals τ and η such that

$$[C_\tau^u; C_\eta^s] \subset C_\alpha$$

and the Hausdorff dimensions of the two sets coincide.

Proof of Theorem 4.1. Let us first study the case $\alpha \in]\alpha_1; \alpha_2[$.

Let $\beta \in \mathbb{R}$ such that $\beta = f'(\alpha)$ [and by (2.4.5) and (2.4.7) we have $\alpha = F'(\beta)$], and define the reals $\alpha^u = (F^u)'(\beta)$ and $\alpha^s = (F^s)'(\beta)$. We have from (2.4.5)

$$\alpha = \alpha^u + \alpha^s \qquad \text{and} \qquad f(\alpha) = f^u(\alpha^u) + f^s(\alpha^s) \tag{4.1}$$

We shall relate $f^u(\alpha^u)$ and $f^s(\alpha^s)$ to the Hausdorff dimensions of the Gibbs measures μ_β^u and μ_β^s [See (2.4.2)]. The singularity sets of μ^u and μ^s are denoted C_α^u and C_α^s.

Lemma 4.2. We have

$$\mu_\beta^u(C_{\alpha^u}^u) = 1 \qquad \text{and} \qquad f^u(\alpha^u) = HD(\mu_\beta^u) = HD(C_{\alpha^u}^u)$$

Proof of Lemma 4.2. We have from (2.2.1) and (2.2.2) for any interval U

$$\mu^u(U) \simeq e^{-n(U)h} \qquad \text{and} \qquad |U| \simeq \exp\left\{\sum_{j=0}^{n(U)-1} J^u[g^j(y(U))]\right\}$$

Since the measure μ_β^u is ergodic, we have the following convergence:

$$\frac{\operatorname{Log} \mu^u(U)}{\operatorname{Log}|U|} \sim \frac{h}{[1/n(U)] \sum_{j=0}^{n(U)-1} -J^u[g^j(y(U))]}$$
$$\xrightarrow[(\Rightarrow n(U) \to +\infty)]{|U| \to 0} \frac{h}{\int -J^u \, d\mu_\beta^u} \quad \mu_\beta^u \quad \text{a.e.}$$

and we obtain therefore

$$\mu_\beta^u(C_{\alpha^u}^u) = 1$$

Using (2.4.2), (2.4.3), and (2.4.5), we have

$$\begin{aligned} f^u(\alpha^u) &= \alpha^u \beta - F^u(\beta) = (F^u)'(\beta)\,\beta - F^u(\beta) \\ &= \frac{h}{\int -J^u \, d\mu_\beta^u} \beta - F^u(\beta) \end{aligned} \tag{4.2}$$

Since we have from (2.4.2)

$$P(\tau_\beta) = h_{\mu_\beta^u} - h\beta - F^u(\beta) \int J^u \, d\mu_\beta^u = 0$$

we get

$$\frac{h_{\mu_\beta^u}}{\int -J^u \, d\mu_\beta^u} = \frac{h}{\int -J^u \, d\mu_\beta^u} \beta - F^u(\beta) \tag{4.3}$$

Comparing (4.2) and (4.3), we obtain

$$f^u(\alpha^u) = \frac{h_{\mu_\beta^u}}{\int -J^u \, d\mu_\beta^u} \tag{4.4}$$

Following refs. 14 and 15, we have

$$\frac{h_{\mu_\beta^u}}{\int -J^u \, d\mu_\beta^u} = HD(\mu_\beta^u) = \inf\{HD(A)/\mu_\beta^u(A) = 1\} \tag{4.5}$$

and we obtain therefore the inequality

$$f^u(\alpha^u) \geqslant HD(C^u_{\alpha^u})$$

The proof of the reverse inequality

$$f^u(\alpha^u) \leqslant HD(C^u_{\alpha^u})$$

parallels the proof of (3.4), and this gives lemma 4.2. ▮

Let us define the measure $\mu_\alpha = \mu^u_\beta \times \mu^s_\beta$. We have from (4.1) and Lemma 4.2

$$f(\alpha) = f^u(\alpha^u) + f^s(\alpha^s) = HD(\mu^u_\beta) + HD(\mu^s_\beta) = HD(\mu^u_\beta \times \mu^s_\beta) = HD(\mu_\alpha)$$

We have also

$$[C^u_{\alpha^u}; C^s_{\alpha^s}] \subset C_\alpha$$

with equal Hausdorff dimensions since $f(\alpha) = f^u(\alpha^u) + f^s(\alpha^s)$. We have then shown Theorem 4.1 in the case $\alpha \in \,]\alpha_1; \alpha_2[$.

We prove now that f, f^u, and f^s are defined at their boundaries. Let first study f^u.

Lemma 4.3. The function f^u is defined on $[\alpha^u_1; \alpha^u_2] \subset \mathbb{R}^{+*}$ and there exist g-invariant measures ρ_1 and ρ_2 such that

$$\alpha^u_1 = \frac{h}{\int -J^u \, d\rho_1} \qquad \text{and} \qquad \alpha^u_2 = \frac{h}{\int -J^u \, d\rho_2}$$

$$f^u(\alpha^u_1) = \frac{h_{\rho_1}}{\int -J^u \, d\rho_1} \qquad \text{and} \qquad f^u(\alpha^u_2) = \frac{h_{\rho_2}}{\int -J^u \, d\rho_2}$$

Proof of Lemma 4.3. We have seen that there exists for any real $\alpha \in \,]\alpha_1; \alpha_2[$ a real β such that

$$\alpha = (F^u)'(\beta) = \frac{h}{\int -J^u \, d\mu^u_\beta}$$

Since the function $(F^u)'$ is strictly decreasing, we have from Theorem 2.4.2

$$\alpha^u_1 = \inf_{\beta \in \mathbb{R}} (F^u)'(\beta) = \lim_{\beta \to +\infty} (F^u)'(\beta)$$

$$\alpha^u_2 = \sup_{\beta \in \mathbb{R}} (F^u)'(\beta) = \lim_{\beta \to -\infty} (F^u)'(\beta)$$

Let L be the set $L=\overline{\{\mu^u_\beta/\beta\in\mathbb{R}\}}\subset M_g(\Lambda)$. Since the functional

$$J:\quad \rho\to\frac{h}{\int -J^u\,d\rho}$$

is continuous on the compact L, it achieves its infimum and its supremum. Let us define g-invariant measures ρ_1 (a cluster point of μ^u_β when $\beta\to+\infty$) and ρ_2 (a cluster point of μ^u_β when $\beta\to-\infty$) which satisfy

$$\alpha^u_1=J(\rho_1)=\frac{h}{\int -J^u\,d\rho_1}\qquad\text{and}\qquad \alpha^u_2=J(\rho_2)=\frac{h}{\int -J^u\,d\rho_2}$$

Since the functional

$$K:\quad \rho\to\frac{h_\rho}{\int -J^u\,d\rho}$$

is upper semicontinuous, we have, using (4.2),

$$\begin{aligned}\lim_{\alpha\to\alpha^u_1} f^u(\alpha)&=\lim_{\beta\to+\infty} f^u((F^u)'(\beta))=\lim_{\beta\to+\infty} K(\mu^u_\beta)\leqslant K(\rho_1)\\ \lim_{\alpha\to\alpha^u_2} f^u(\alpha)&=\lim_{\beta\to-\infty} f^u((F^u)'(\beta))=\lim_{\beta\to-\infty} K(\mu^u_\beta)\leqslant K(\rho_2)\end{aligned}\tag{4.6}$$

Now we apply the variational principle (2.4.1) and (2.4.2) to the measures ρ_1 and ρ_2, and we get, for example, for large $\beta>0$

$$h_{\rho_1}-h\beta-F^u(\beta)\int J^u\,d\rho_1<0$$

which gives with the values of α^u_1 and (2.4.6)

$$K(\rho_1)\leqslant\alpha^u_1\beta-F^u(\beta)\leqslant(F^u)'(\beta)\,\beta-F^u(\beta)=f^u((F^u)'(\beta))$$

Taking the limit when β goes to $+\infty$, we get

$$K(\rho_1)\leqslant\lim_{\beta\to+\infty} f^u((F^u)'(\beta))=\lim_{\alpha\to\alpha^u_1} f^u(\alpha)\tag{4.7}$$

Comparing (4.4) and (4.7), we obtain

$$\lim_{\alpha\to\alpha^u_1} f^u(\alpha)=f^u(\alpha^u_1)=K(\rho_1)$$

By the same method we have for large $\beta<0$

$$K(\rho_2)\leqslant\alpha^u_2\beta-F^u(\beta)\leqslant(F^u)'(\beta)\,\beta-F^u(\beta)=f^u((F^u)'(\beta))$$

which leads to the inequality

$$K(\rho_2) \leqslant \lim_{\beta \to -\infty} f^u((F^u)'(\beta)) = \lim_{\alpha \to \alpha_2^u} f^u(\alpha)$$

We get therefore

$$\lim_{\alpha \to \alpha_2^u} f^u(\alpha) = f^u(\alpha_2^u) = K(\rho_2)$$

which gives Lemma 4.3. ∎

These results can be extended obviously to the function f^s and, as we shall see later, to the function f. We can find therefore g-invariant measures ξ_1 and ξ_2 such that

$$\alpha_1^s = \frac{h}{\int -J^u \, d\xi_1} \qquad \text{and} \qquad \alpha_2^s = \frac{h}{\int -J^u \, d\xi_2}$$

$$f^s(\alpha_1^s) = \frac{h_{\xi_1}}{\int -J^u \, d\xi_1} \qquad \text{and} \qquad f^s(\alpha_2^s) = f^s(\alpha_2^s) = \frac{h_{\xi_2}}{\int -J^u \, d\xi_2}$$

Let us define the measures $\zeta_1 = \rho_1 \times \xi_1$ and $\zeta_2 = \rho_2 \times \xi_2$; we have from (4.1)

$$\alpha_1 = \alpha_1^u + \alpha_1^s \qquad \text{and} \qquad \alpha_2 = \alpha_2^u + \alpha_2^s$$

$$f(\alpha_1) = f^u(\alpha_1^u) + f^s(\alpha_1^s) = HD(\zeta_1)$$

and

$$f(\alpha_2) = f^u(\alpha_2^u) + f^s(\alpha_2^s) = HD(\zeta_2)$$

which gives Theorem 4.1. ∎

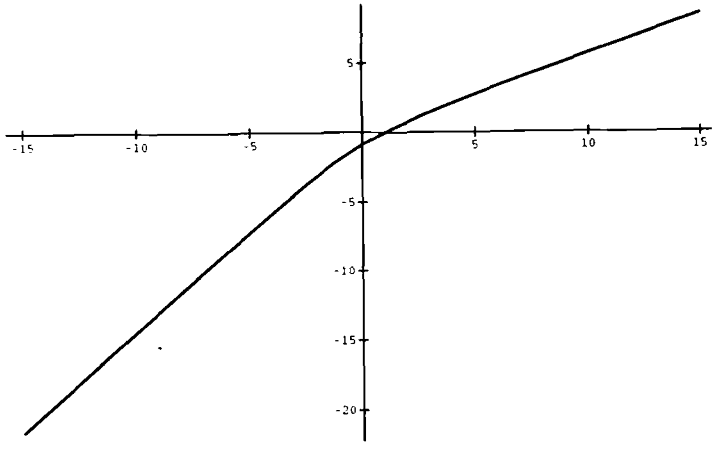


Fig. 1. Graph of the free energy function F^s, $F^s\colon \mathbb{R} \to \mathbb{R}$, $\beta \to F^s(\beta)$.

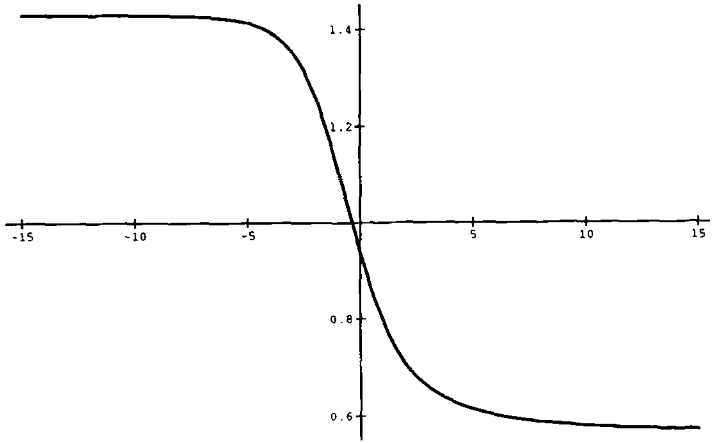


Fig. 2. Graph of the derivative of the free energy function F^s, $(F^s)': \mathbb{R} \to]\alpha_1^s; \alpha_2^s[$, $\beta \to (F^s)'(\beta)$.

We have then shown that in the general case the dimensional spectrum $f(\alpha)$ is defined on an interval $[\alpha_1; \alpha_2] \subset \mathbb{R}^{+*}$ and has the typical concave shape.

Remarks. 1. There are some remarkable values: $F^u(0) = 1$; $F^u(1) = 0$ and $f^u((F^u)'(0)) = 1$; $f^s((F^s)'(0)) = \delta \leqslant 1$ and $\delta = 1$ if and only if the Gibbs measure of J^s, μ_{J^s}, is absolutely continuous with respect to the Lebesgue measure.[1]

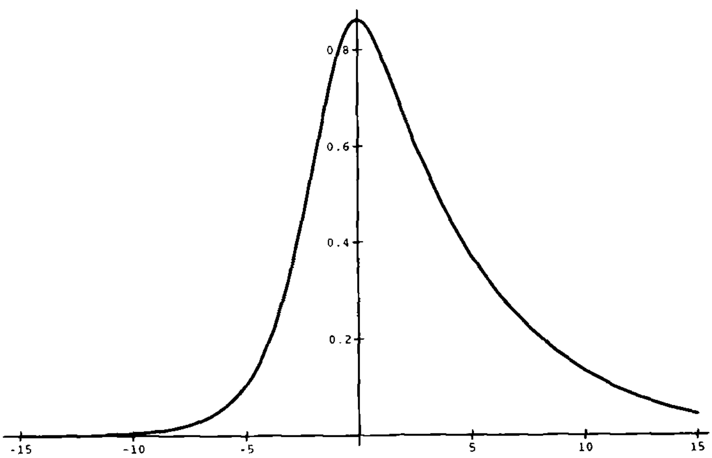


Fig. 3. Graph of a tranform of the dimension spectrum f^s, $H: \mathbb{R} \to]\inf(f^s(\alpha_1^s); f^s(\alpha_2^s)); -F^s(0)]$, $\beta \to \beta(F^s)'(\beta) - F^s(\beta) = f^s((F^s)'(\beta))$.

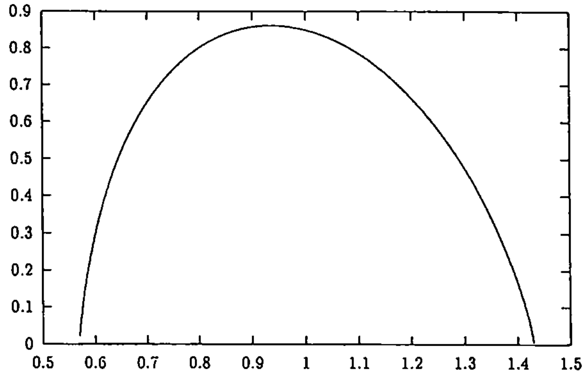


Fig. 4. Graph of the dimension spectrum f^s, $f^s\colon [\alpha_1^s; \alpha_2^s] \to [\inf(f^s(\alpha_1^s; \alpha_2^s)); \; -F^s(0)]$, $\alpha \to f^s(\alpha)$.

2. For any real β, the line $y = \beta\alpha - F^u(\beta)$ is tangent to the graph $\alpha \to f(\alpha)$ at the point $\alpha = (F^u)'(\beta)$.

3. The function f is degenerate if and only if the functions f^u and f^s are degenerate. In this case, if f takes the value 2, then the measures μ^u and μ^s, and consequently μ, are equivalent to the Lebesgue measure.

To complete this study we give the graph of the different functions we have found (this is just for the illustration of the general shape): the free energy function F^s (Fig. 1), the derivative $F^{s\prime}$ of the free energy function (Fig. 2), the function H which is a transform of the dimension spectrum [see (2.4.6)] (Fig. 3), and the dimension spectrum function f^s (Fig. 4).

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